Application of Gradient Expansion to Inflationary Universe

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(1 Nobember 1994)

Abstract

Using the long wave perturbation scheme(gradient expansion), the effect of inhomogeneity on the inflationary phase is investigated. We solved the perturbation equation of which source term comes from inhomogeneity of a scalar field and a seed metric. The result indicates that sub-horizon scale inhomogeneity strongly affects the onset of inflation.

I. INTRODUCTION

Dynamics of inhomogeneity plays an important role in many phenomena in general relativity. Generality of inflation is one of such a problem. To explain the present our universe, we must have an inflationary phase in past. But as we do not know the initial condition of universe, we hope that universe can enter inflationary phase from wide range of initial condition. For homogeneous closed universe, the universe cannot enter inflationary phase if the initial curvature is too large [1,2]. For inhomogeneous universe, the effect of strong(non-linear) initial inhomogeneity is studied for space-time with symmetry by numerical method [7]. These result indicate that sub-horizon scale inhomogeneity strongly prevent the onset of inflationary phase. But imposing symmetry to space-time may lose general feature of dynamics of inhomogeneity. We want to know about more general situation without symmetry using appropriate approximation that include non-linear effect. Gradient expansion(GE) [3–6] is one of such a method. This approximation scheme expands the Einstein

equation by the number of the spatial gradient. As the background solution, we have an inhomogeneous space-time that does not include the effect of a spatial curvature. We can include the curvature by calculating next order. This method describes a long-wave non-linear perturbation and it is suitable to applying to the inflationary universe. Furthermore, we can treat non-linear effect without imposing any symmetry for a space-time.

In this paper, we treat Einstein gravity with a minimally coupled scalar field by GE. Our purpose is to observe the effect of large scale inhomogeneity on inflationary phase. Especially, we are interested in how the onset of inflation is affected by the initial inhomogeneity.

This paper is organized as follows. In Sec.II, we shortly review GE approximation using Hamilton-Jacobi method. We derive the second order equation of motion that include the scalar field inhomogeneity. In Sec.III, we obtain the solution for the inflationary phase and discuss the effect of inhomogeneity on inflation. Sec.IV is devoted to summary.

II. EQUATION OF MOTION BY H-J METHOD

We follow the method of Salopek [5] and derive the second order equation of motion of GE by Hamilton-Jacobi(H-J) method. H-J equation is

$$\begin{aligned}\mathcal{H} &= 2\kappa\gamma^{-1/2}\frac{\delta S}{\delta\gamma_{ij}}\frac{\delta S}{\delta\gamma_{kl}}(\gamma_{jk}\gamma_{il}-\frac{1}{2}\gamma_{ij}\gamma_{kl})-\frac{1}{2\kappa}\gamma^{1/2}R+\frac{1}{2}\gamma^{-1/2}(\frac{\delta S}{\delta\phi})^2+\gamma^{1/2}V(\phi)=0,\\ \mathcal{H}_i &= -2(\gamma_{ik}\frac{\delta S}{\delta\gamma_{kj}})_{,j}+\frac{\delta S}{\delta\gamma_{lk}}\gamma_{lk,i}+\frac{\delta S}{\delta\phi}\phi_{,i}=0,\end{aligned} \tag{1}$$

where $\kappa = 8\pi G$. These two equations are Hamiltonian constraint and momentum constraint, respectively. We expand the generating functional S by the number of spatial derivative

$$S = S^{(0)} + S^{(2)} + \cdots.$$

The form of generating functional $S^{(0,2)}$ is assumed to be

$$S^{(0)} = -\frac{2}{\kappa}\int d^3x\gamma^{1/2}H(\phi),\ \ S^{(2)} = \int d^3x\gamma^{1/2}(J(\phi)R + K(\phi)\phi_{;i}\phi^{;i}), \tag{2}$$

where ; represents spatial covariant derivative associated with γ_{ij}. This form of generating functional satisfies momentum constraint strongly provided that parameter fields(integration constant of H-J equation) are spatially constant.

The zeroth order Hamiltonian becomes

$$\mathcal{H}^{(0)} = -\frac{3}{\kappa}\gamma^{1/2}\left[H^2 - \frac{2}{3\kappa}\left(\frac{\partial H}{\partial \phi}\right)^2 - \frac{\kappa}{3}V(\phi)\right],$$

and the requirement that it vanishes gives

$$H^2 = \frac{2}{3\kappa}\left(\frac{\partial H}{\partial \phi}\right)^2 + \frac{\kappa}{3}V(\phi) \tag{3}$$

The second order Hamiltonian becomes

$$\begin{aligned}\mathcal{H}^{(2)} = \gamma^{1/2}R\left[HJ - \frac{1}{2\kappa} - \frac{2}{\kappa}\frac{\partial H}{\partial \phi}\frac{\partial J}{\partial \phi}\right] + \gamma^{1/2}\phi_{;m}\phi^{;m}\left[HK + \frac{1}{2} + \frac{2}{\kappa}\frac{\partial H}{\partial \phi}\frac{\partial K}{\partial \phi} - 4H\frac{\partial^2 J}{\partial \phi^2}\right] \\ + \gamma^{1/2}\phi_{;m}^{\ ;m}\left[\frac{4}{\kappa}\frac{\partial H}{\partial \phi}K - 4H\frac{\partial J}{\partial \phi}\right].\end{aligned} \tag{4}$$

The requirement that this vanishes gives the following equations for J, K:

$$\begin{aligned} HJ - \frac{1}{2\kappa} - \frac{2}{\kappa}H'J' = 0, \\ \frac{1}{\kappa}H'K - HJ' = 0, \\ HK + \frac{1}{2} + \frac{2}{\kappa}H'K' - 4HJ'' = 0, \end{aligned} \tag{5}$$

where $' = \frac{\partial}{\partial \phi}$. We have two equations for K, but it can be shown that the second equation automatically satisfies the third one. So they are consistent.

Now the evolution equation that is accurate to the second order spatial gradient becomes the following form. From now on, we use the synchronous coordinate system $N = 1, N_i = 0$:

$$\begin{aligned}\dot{\phi} &= \gamma^{-1/2}\frac{\delta S}{\delta \phi} \\ &= -\frac{2}{\kappa}\frac{\partial H}{\partial \phi} + \frac{\partial J}{\partial \phi}R(\gamma) - \left[\frac{\partial K}{\partial \phi}\phi_{;m}\phi^{;m} + 2K\phi_{;m}^{\ ;m}\right],\end{aligned} \tag{6}$$

$$\begin{aligned}\dot{\gamma}_{ij} &= 4\kappa\gamma^{-1/2}(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl})\frac{\delta S}{\delta \gamma_{kl}} \\ &= 4\kappa(\frac{1}{2\kappa}H\gamma_{ij} + K(\frac{1}{4}\gamma_{ij}\phi_{;m}\phi^{;m} - \phi_{;i}\phi_{;j}) \\ &\quad + J(\frac{1}{4}\gamma_{ij}R(\gamma) - R_{ij}(\gamma)) + J_{;ij})\end{aligned} \tag{7}$$

The first order solution is obtained by integrating the following equation:

$$
\begin{aligned}
\dot{\phi} &= -\frac{2}{\kappa}H'(\phi), \\
\dot{\gamma}_{ij} &= 2H\gamma_{ij}, \qquad (8)
\end{aligned}
$$

and the solution becomes

$$
\gamma_{ij}^{(0)} = A(\phi)h_{ij}(x), \quad \phi^{(0)} = \phi_0(t - t_0(x)),
$$

where $h_{ij}(x)$ is an arbitrary spatial function(seed metric) and $A^{1/2} = \int dtH$. t_0 is a integration constant that depends only on spatial coordinate, and a scale factor A depends on spatial coordinate through ϕ. We can rewrite above equations (6), (7) using t_0 and covariant derivative associated with a seed metric h_{ij}. The spatial curvature becomes

$$
\begin{aligned}
R_{ij}(\gamma^{(0)}) &= R_{ij}(h) - h_{ij}\left[(\dot{H} + H^2)t_{0|m}t_0{}^{|m} - H\Delta t_0\right] \\
&\quad + (-\dot{H} + H^2)t_{0|i}t_{0|j} + Ht_{0|ij}, \qquad (9) \\
R(\gamma^{(0)}) &= \frac{1}{A}(R(h) + (-4\dot{H} - 2H^2)t_{0|m}t_0{}^{|m} + 4H\Delta t_0),
\end{aligned}
$$

where $t_{0|ij}$ means covariant derivative associate with h_{ij}. The derivative of ϕ, J is

$$
\phi_{;ij} = \frac{2}{\kappa}H't_{0|ij} - \frac{2}{\kappa}HH'h_{ij}t_{0|m}t_0{}^{|m} + \left(\frac{2}{\kappa}\right)^2 (H''H' + \kappa HH')t_{0|i}t_{0|j}, \qquad (10)
$$

$$
\begin{aligned}
4\kappa J_{;ij} &= 8\left[((H')^2 + \frac{3}{2}\kappa H^2)J - \frac{3}{4}H\right]t_{0|i}t_{0|j} + 8(\frac{\kappa}{2}HJ - \frac{1}{4})t_{0|ij} \\
&\quad - 8H(\frac{\kappa}{2}HJ - \frac{1}{4})h_{ij}t_{0|m}t_0{}^{|m}. \qquad (11)
\end{aligned}
$$

To derive these expression, we have used eq.(5). Using these relations, the form of evolution equation becomes

$$
\dot{\phi} = -\frac{2}{\kappa}H' + \frac{J'}{A}R(h) - \frac{1}{\kappa}\frac{H'}{A}t_{0|m}t_0{}^{|m}, \qquad (12)
$$

$$
\begin{aligned}
\dot{\gamma}_{ij} &= 2H\gamma_{ij} + 4\kappa J(\frac{1}{4}R(h)h_{ij} - R_{ij}(h)) \qquad (13) \\
&\quad + 2H(-t_{0|i}t_{0|j} + \frac{1}{2}h_{ij}t_{0|m}t_0{}^{|m}) - 2t_{0|ij}.
\end{aligned}
$$

We consider the perturbation due to the second order spatial gradient. By linearizing eq.(12) and (13), we get

$$\delta\dot{\phi} = -\frac{2}{\kappa}H''\delta\phi + E(\phi_0, h), \tag{14}$$

$$\delta\dot{h}_{ij} = 2H'h_{ij}\delta\phi + F_{ij}(\phi_0, h), \tag{15}$$

where $\delta h_{ij} = \delta\gamma_{ij}/A$ and E, F_{ij} are source terms from the second order gradient:

$$E = \frac{J'}{A}R(h) - \frac{1}{\kappa}\frac{H'}{A}t_{0|m}t_0{}^{|m},$$
$$F_{ij} = 4\kappa\frac{J}{A}(\frac{1}{4}R(h)h_{ij} - R_{ij}(h)) + 2\frac{H}{A}(-t_{0|i}t_{0|j} + \frac{1}{2}h_{ij}t_{0|m}t_0{}^{|m}) - \frac{2}{A}t_{0|ij}.$$

These terms represent non-linear inhomogeneity due to the seed metric h_{ij} and the zeroth order matter field $\phi(t - t_0(x))$. The solution of (14) becomes

$$\delta\phi = \left(\int_{ti}^{t} dt\frac{E}{\dot{\phi}_0}\right)\dot{\phi}_0, \tag{16}$$

where we have chosen the integration constant such as $\delta\phi(t_i) = 0$. Using this solution, the metric perturbation becomes

$$\begin{aligned}\delta h_{ij} &= \int_{ti}^{t}(2H'\delta\phi h_{ij} + F_{ij})\\ &= 2H\int_{ti}^{t} dt\frac{E}{\dot{\phi}_0}h_{ij} + \int_{ti}^{t} dt\left(-2H\frac{E}{\dot{\phi}_0}h_{ij} + F_{ij}\right).\end{aligned} \tag{17}$$

III. SOLUTION IN THE INFLATIONARY PHASE

We derive the solution in the inflationary phase $\ddot{a} > 0$. We first evaluate the function J:

$$\begin{aligned}J &= \frac{1}{2\kappa a}\left(\int dta + const\right)\\ &= \frac{1}{2\kappa}\{H^{-1} - H^{-1}(H^{-1})^{\cdot} + H^{-1}(H^{-1}(H^{-1})^{\cdot})^{\cdot} - \cdots\\ &\qquad - \frac{1}{a}\left(a_iH_i^{-1} - a_iH_i^{-1}(H_i^{-1})^{\cdot} + \cdots\right) + \frac{const}{a}\},\end{aligned} \tag{18}$$

and we choose the integral constant so that the term proportional to $1/a$ vanishes.

$$J = \frac{1}{2\kappa}H^{-1}\left\{1 - (H^{-1})^{\cdot} + (H^{-1}(H^{-1})^{\cdot})^{\cdot} - \cdots\right\}. \tag{19}$$

During inflation, we can neglect all but the first term in this expression because the condition $|\dot{H}| < H^2$ is satisfied. To the leading order of this expansion, we can evaluate the following integrals:

$$\int_{ti}^{t} dt \frac{J'}{\dot{\phi}_0 A} \approx -\frac{1}{8}\left[a^{-2}H^{-3}\right]_{ti}^{t},$$
$$\int_{ti}^{t} dt \frac{H'}{\dot{\phi}_0 A} \approx \frac{\kappa}{4}\left[a^{-2}H^{-1}\right]_{ti}^{t},$$

and the perturbation of the scalar field becomes

$$\frac{\delta\phi}{\dot{\phi}_0} = -\frac{1}{8}R(h)\left[a^{-2}H^{-3}\right]_{ti}^{t} - \frac{1}{4}t_{0|m}t_0{}^{|m}\left[a^{-2}H^{-1}\right]_{ti}^{t}. \tag{20}$$

Therefore during inflation, $\delta\phi/\dot{\phi}_0$ approaches a constant value and $\delta\dot{\phi}$ goes to zero in a Hubble time scale:

$$\frac{\delta\phi}{\dot{\phi}_0} \approx \left(\frac{1}{8}R(h)(a^{-2}H^{-3})_{ti} + \frac{1}{4}t_{0|m}t_0{}^{|m}(a^{-2}H^{-1})_{ti}\right) \equiv \left(\frac{\delta\phi}{\dot{\phi}_0}\right)_0, \quad \delta\dot{\phi} \approx 0. \tag{21}$$

The metric perturbation during inflation is expressed as

$$\delta h_{ij} = 2H\left(\frac{\delta\phi}{\dot{\phi}_0}\right)_0 h_{ij} - (a^{-2}H^{-2})_{ti}R_{ij}(h) - (a^{-2})_{ti}t_{0|i}t_{0|j} - (a^{-2}H^{-1})_{ti}t_{0|ij}. \tag{22}$$

The metric that is not proportional to h_{ij} approaches to constant values in a Hubble time scale. GE is correct if $|\phi_0| > |\delta\phi|$ and $|h_{ij}| > |\delta h_{ij}|$ are satisfied. These conditions are rewritten to be

$$\left|\frac{R(h)}{2a^2H^2} + \frac{t_{0|m}t_0{}^{|m}}{a^2}\right| < 2,$$
$$|R(\gamma^{(0)})| < |R_*| \equiv 12H^2. \tag{23}$$

They are well satisfied during inflation provided that they are satisfied at onset of inflation because the quantities in the left hand side of the condition decrease in time during inflation. Therefore the expression of perturbation (21), (22) are correct in whole time of inflation.

A. Effect of the spatial curvature on inflation

We investigate the effect of the curvature inhomogeneity. For this purpose, let us consider a local Hubble radius and a local scale factor that are defined by

$$\tilde{H} = \frac{1}{6}\gamma^{ij}\dot{\gamma}_{ij},$$
$$\tilde{a} = \exp(\int dt\tilde{H}). \qquad (24)$$

Using eq.(12), these quantity become

$$\tilde{H} = H\left(1 - \frac{1}{12H^2}R(\gamma^{(0)})\right),,$$
$$\tilde{a} = a\left(1 + \frac{1}{24}\left[\frac{R(\gamma^{(0)})}{H^2}\right]_{ti}^{t}\right) \qquad (25)$$

The co-moving Hubble radius becomes

$$(\tilde{a}\tilde{H})^{-1} = (aH)^{-1}\left[1 - \frac{1}{24}\left(\frac{R(\gamma^{(0)})}{H^2} + (\frac{R(\gamma^{(0)})}{H^2})_{ti}\right)\right]^{-1}. \qquad (26)$$

This quantity decreases during inflation and grows in non-inflationary phase. We choose t_i, t_f are onset and end time of inflation, respectively. At $t = t_i$, $(aH)^{-1}$ has maximum and at $t = t_f$ minimum. The maximum value at $t = t_i$ for $R(\gamma^{(0)}) < 0$ becomes smaller than one for $R(\gamma^{(0)}) = 0$ and larger for $R(\gamma^{(0)}) > 0$. This implies that the negative $R(\gamma^{(0)})$ enhances and the positive $R(\gamma^{(0)})$ suppresses inflatinary phase. Especially if $R(\gamma^{(0)})$ becomes as large as $R_* = 12H_{ti}^2$, the co-moving Hubble radius goes to infinity at $t = t_i$ and inflation may not occur. Of course we cannot apply GE to such a regime because the condition (23) is violated. But we extrapolate the behavior for small $|R|$ and predict the behavior for large $|R|$. As the curvature radius is defined by $L_R = \sqrt{6/R}$, this critical value corresponds to the scale $L_R = \sqrt{2}H_i^{-1}$. If the scale of curvature inhomogeneity is larger than this scale, inflation may occur. But the scale is smaller than this value, we may not have inflation. This result is consistent with the result of the calculation of numerical relativity [7].

To see the effect of curvature from a different view point, we consider local e-folding time of inflation:

$$\tilde{N}(t_i \to t_f) = \int_{ti}^{tf} dt\tilde{H} \approx N + \frac{1}{24}\left[\frac{R(\gamma^{(0)})}{H^2}\right]_{ti}^{tf} \approx N - \frac{1}{24}(\frac{R(\gamma^{(0)})}{H^2})_i, \qquad (27)$$

where N is the e-folding for $R(\gamma^{(0)}) = 0$. Positive R makes e-folding shorter and negative R makes it longer than $R = 0$ case. This also indicates that positive spatial curvature suppresses inflationary phase.

We check the strong energy condition:

$$\rho + \sum p_i = -\frac{6}{\kappa} H^2 \left(1 + \frac{\dot{H}}{H^2} (1 + \frac{R(h)}{12(a^2 H^2)} + \frac{1}{6a^2} t_{0|m} t_0{}^{|m}) \right). \tag{28}$$

At the onset of inflation, $(\dot{H}/H^2)_{ti} = -1$, so $(\rho + \sum p_i)_{ti} = H^2/\kappa(R(h)/(2a^2 H^2)_{ti} + t_{0|m} t_0{}^{|m}/(a^2)_{ti})$. Positive $R(h)$ or $t_{0|m} t_0{}^{|m}$ term make $(\rho + \sum p_i) > 0$ and the strong energy condition is satisfied. This means that inflation does not occur. Negative $R(h)$ can make $(\rho + \sum p_i)_{ti} < 0$ if its value is larger than the $t_{0|m} t_0{}^{|m}$ term, and the strong energy condition is violated.

B. Numerical calculation

To check the result of the previous subsection, we solve eq.(13),(14) numerically. We prepare the initial condition at pre-inflationary phase. For simplicity, we consider only $t_0 = 0$ case. The model is $V(\phi) = 1/2 m^2 \phi^2$. Parameter and initial value is $m^2 = 0.5, \phi(0) = 7, \dot{\phi}(0) = -5, J(0) = 0$ in $\kappa = 1$ unit. The scalar curvature $R(h) = -30, 0, 30$. This value is chosen to satisfy the condition (23). Fig.1 is a phase space diagram $(\phi, \dot{\phi})$. The solid line is $R(h) = 0$, dotted line is $R(h) < 0$ and dashed line is $R(h) > 0$. The zeroth order solution($R(h) = 0$) has inflationary phase from $\phi \approx 6$ to $\phi \approx 1$. For positive $R(h)$, the trajectory enters inflationary phase with smaller ϕ value(smaller H). For negative $R(h)$, the trajectory enters inflation with larger ϕ value(larger H). After entering inflation, all trajectories approaches attractor solution. Fig.2 is time evolution of the perturbation of scalar field $\delta\phi$. In Hubble time scale, its value approaches a constant value. Fig.3 shows the co-moving Hubble radius. $R(h) = 0$ curve has a maximum at $t \approx 0.06$ and after this time, the system enters inflation. The positive $R(h)$ makes the co-moving Hubble radius larger and the negative $R(h)$ makes smaller. These behavior is consistent with the analysis of the previous subsection.

IV. SUMMARY

As we have shown, GE becomes good approximation in the inflationary phase because the perturbation goes to a constant value. If the initial inhomogeneity is not so large as to break the approximation, we can use GE as a model to construct an inhomogeneous inflationary space-time. As is well known, a global structure of inflation is very complicated because of continuous creation of new inflationary domain via quantum fluctuation. We can grasp these feature by using a stochastic approach [8–10], but this method does not incorporate the information of geometry completely. GE can describe the space-time of inflation whose local Hubble radius is different from place to place. The zeroth order solution has the form $\phi = \phi_0(t - t_0(x))$ and choosing the time delay function $t_0(x)$ appropriately, we can construct such a space-time as the solution of GE. We will discuss on this topics in a separated publication [11].

Note: After this work has completed, we noticed that the paper [12]. They treat this subject in the same context as ours. But they perform only numerical analysis and does not derive analytic expression of perturbation.

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FIGURES

FIG. 1. A phase space diagram $(\phi, \dot{\phi})$. for $R(h) = 0$(solid line), $R(h) > 0$(dotted line) and $R(h) < 0$(dashed line). $R(h) = 0$ solution enters inflation at $\phi \approx 6$. For positive $R(h)$, solution enters inflation with smaller ϕ value and for negative $R(h)$, solution enters inflation with larger ϕ value.

FIG. 2. Time evolution of perturbation $\delta\phi$. Perturbation approaches constant value in Hubble time scale.

FIG. 3. Time evolution of co-moving Hubble radius. System enters inflationary phase at $t \approx 0.06$ for $R(h) = 0$ case. For positive $R(h)$, co-moving Hubble radius at the onset of inflation becomes larger. For negative $R(h)$, co-moving Hubble radius at the onset of inflation becomes smaller.

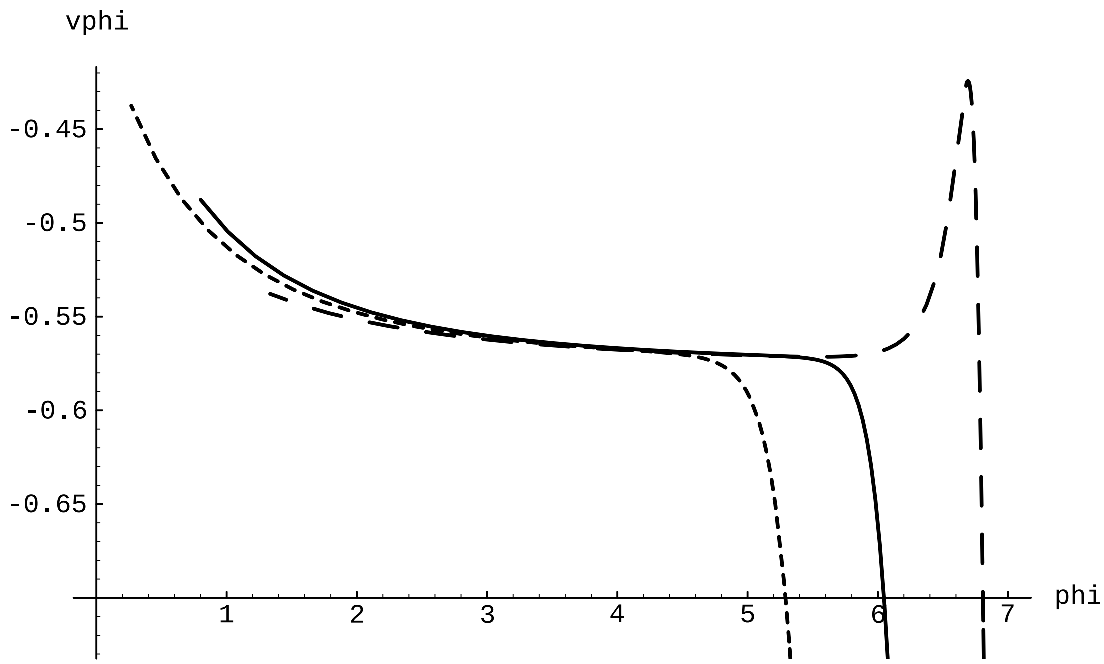

Fig.1

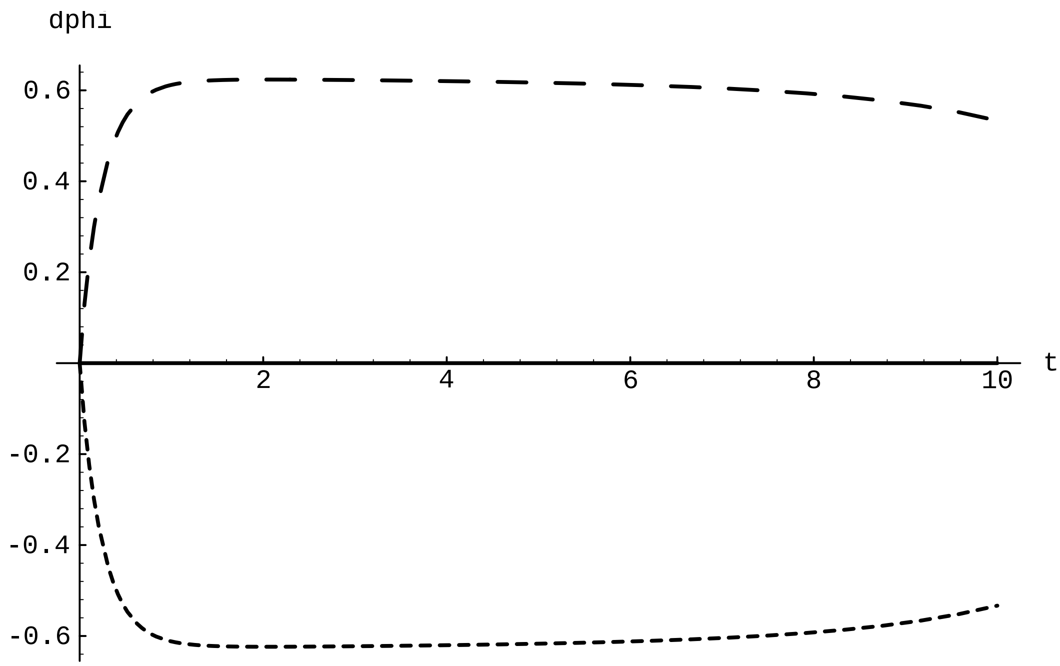

Fig.2

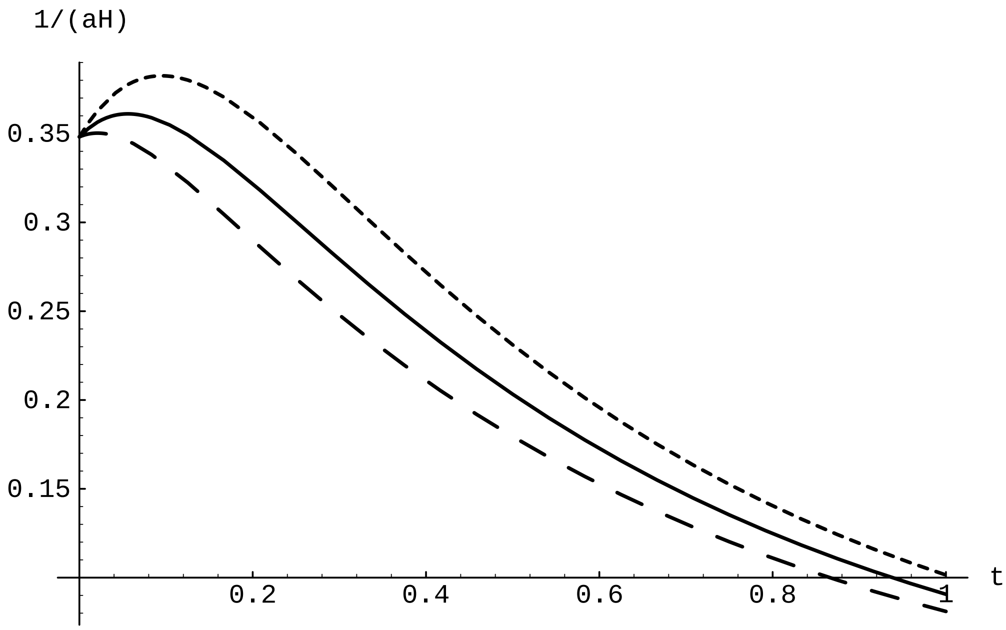

Fig.3